Math 246A Lecture 2 Notes

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September 28, 2018

1 Real and Complex Derivatives

1.1 The fundamental theorem of algebra (cont.)

Proof. Last time, we had $p(z) = b_0 + b_k(z - z_0)^k + \sum_{j=k+1}^n b_j(z - z_0)^j$, where $p(z) \geq b_0$ for all $z \in \mathbb{C}$. Choose z such that $|z - z_0| = \delta > 0$. Let

$$|R(z)| \leq \delta^{k+1} \sum_{j=k+1}^{n} |b_j| \leq \delta \frac{\sum_{j=K+1}^n |b_j|}{|b_k|} |Q(z)| \leq \frac{1}{2}|Q(z)|.$$

Since $(z - z_0)^k$ maps the circle C_δ to a circle it wraps around at least once, we can pick a z such that $Q(z)$ is the closest point on the image circle to the origin. So $Q(z) = -\frac{b_0}{|b_0|}|b_k|\delta^k$, and $z = (z - z_0)^k \delta^k e^{ik\alpha}$. Then

$$|b_0 + Q(z)| = \left| b_0 - \frac{b_0}{|b_0|}|b_k|\delta^k \right| = |b_0| \left(1 - \frac{|b_k|}{|b_0|}\delta^k\right) < |b_0|,$$

which is a contradiction. $\square$

1.2 Derivatives

Let Ω be an open subset of $\mathbb{R}^2 = \mathbb{C}$, and let $f : \Omega \to \mathbb{R}^2$ be

$$f(x, y) = \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix}.$$

Definition 1.1. f is **differentiable** at (x_0, y_0) if there exists a 2×2 real matrix A such that

$$f(x, y) = f(x_0, y_0) + A \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} + o(|z - z_0|).$$

Here,

$$A = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}.$$

This condition is equivalent to

$$\frac{f(x,y) - \left(f(x_0,y_0) + A\begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix}\right)}{|z-z_0|} \xrightarrow{z\to z_0} 0.$$

What can A do to circles? If $\det(A) = 0$, A can send a circle to a line. Otherwise, A sends circles to circles. Depending on the sign of the determinant of A, it can send circles (going counterclockwise) to circles (going clockwise).

Definition 1.2. $f = u(x,y) + iv(x,y)$ is **complex differentiable** at $z_0 = x_0 + iy_0$ if there exists some $z_0 \in \mathbb{C}$. such that $f(z) = f(z_0) + f'(z_0)(z-z_0) + o(|z-z_0|)$; i.e.

$$\frac{f(z) - (f(z_0) + f'(z_0)(z-z_0))}{z - z_o} \xrightarrow{z\to z_0} 0.$$

Let A be the 2×2 real matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Definition 1.3. A is **complex linear** if there exist $\alpha, \beta \in \mathbb{R}^2$ such that

$$A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha x - \beta y \\ \beta x + \alpha y \end{bmatrix}.$$

Lemma 1.1. *The following are equivalent:*

1. *A is complex linear.*
2. *If* $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $AJ = JA$.
3. $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$ *for some* $(\alpha, \beta) \in \mathbb{R}^2$.
4. $A = 0$ *or*

$$A = \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

   for some $R > 0$, $\theta \in [0, 2\pi)$.

5. *If* $\det(A) \neq 0$, *A is conformal (angle preserving and orientation preserving).*

Proof. Most of the equivalences are clear. We show $4 \implies 5$. Let $A \neq 0$. Then

$$A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}\begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$ □

Example 1.1. Look at the function

$$f(z) = \overline{z} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

The matrix is

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

This is not complex linear, and f is not complex differentiable because it does not preserve orientation; it sends a clockwise circle to a counterclockwise one (and vice versa).

Proposition 1.1. *Let $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ with $a_j \in \mathbb{C}$ and $\Omega = \mathbb{C}$. Then P is complex differentiable at z_0 for all z_0, and*

$$p'(z_0) = \sum_{j=1}^{n} j a_j z^{j-1}.$$

Proof. We proceed from the definition.

$$\begin{aligned} p(z) - p(z_0) &= a_1(z - z_0) + \sum_{j=2}^{n} a_j (z^j - z_0^j) \\ &= a_1(z - z_0) + \sum_{j=2}^{n} a_j \left(\sum_{k=0}^{j-1} z^k z_0^{j-1-k} \right) \end{aligned}$$

So we get that

$$\frac{p(z) - p(z_0)}{z - z_0} \xrightarrow{z \to z_0} a_1 + \sum_{j=2}^{n} j a_j z_0^{j-1}. \qquad \square$$

Theorem 1.1 (Cauchy-Riemann equations)**.** *Let $f : \Omega \to \mathbb{R}^2$ be differentiable at (x_0, y_0), so*

$$f(x, y) = f(x_0, y_0) + A \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} + o(|z - z_0|), \quad \text{where } A = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}.$$

Then f is complex differentiable at z_0 iff $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$.

1.3 Power series

Let S be any set, and let $a : S \to \mathbb{C}$.

Definition 1.4. We define the **sum** over S of a to be

$$\sum_{S} |a| = \sup_{\substack{F \subseteq S \\ F \text{ finite}}} \sum_{F} |a|.$$

Lemma 1.2. *The following are true about sums:*

1. *Suppose* $a_n \geq 0$ *and* $n \in \mathbb{N}$. *Then* $\sum_{n=1}^{\infty} a_n < \infty \iff \sum_{\mathbb{N}} |a_n| < \infty$. *In fact, these are equal. Also*
$$\sum a_n = \sum a_{j(n)}$$
if $n \mapsto j(n)$ *is a bijective map from* $\mathbb{N} \to \mathbb{N}$.

2. *Suppose* $a_{j,k} \in \mathbb{C}$ *for* $j, k \in \mathbb{N}$. *Then*
$$\sum_{\mathbb{N}\times\mathbb{N}} |a_{j,k}| < \infty \implies \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{j,k} = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} a_{j,k}$$
and both converge absolutely.